We consider a two-particle/two-setting Bell experiment to visualize the conflict between Bell–Zukowski inequality and Bell–Mermin inequality. The experiment is reproducible by local realistic theories which are not rotationally invariant. We find that the average value of Bell–Zukowski operator is evaluated by the two-particle/two-setting Bell experiment in question. Bell–Zukowski inequality reveals that the constructed local realistic theories for the experiment are not rotationally invariant. The two-particle Bell experiment in question reveals the conflict between Bell–Zukowski inequality and Bell–Mermin inequality. Our analysis finds the threshold visibility for the two-particle interference to reveal the conflict noted above. The threshold visibility agrees with the value to obtain a violation of Bell–Zukowski inequality.

Keywords: The quantum theory; local realism.

PACS Nos.: 03.65.Ud, 03.67.Mn

1. Introduction

Local and realistic theories assume that physical properties exist irrespective of whether they are measured and that the result of measurement pertaining to one system is independent of any other measurement simultaneously performed on a different system at a distance. As Bell reports, a certain inequality that correlation functions of a local realistic theory obey is violated by the quantum predictions. Bell uses the singlet state to demonstrate this. Likewise, a certain set of correlation functions produced by the quantum predictions in a single quantum state contradicts local realistic theories. We see the conflict between local realism and the quantum predictions. Since Bell’s work, local realistic theories are researched extensively. Numerous experiments show that Bell inequalities and local realistic theories are violated.

In 1982, Fine presents the following example. A set of correlation functions is described with the property that they are reproducible by local realistic theories for a system in two-partite states if and only if the set of correlation functions
satisfies the complete set of (two-setting) Bell inequalities. This is generalized to a system described by multipartite\textsuperscript{15-17} states in the case where two dichotomic observables are measured per site. We therefore obtain a necessary and sufficient condition for a set of correlation functions to be reproducible by local realistic theories in the specific case mentioned above.

A violation of "standard" two-setting Bell inequalities\textsuperscript{15-20} is sufficient for experimentalists to show the conflict between local realism and the quantum predictions. However, it is necessary to create an entangled state with sufficient visibility to violate a Bell inequality. Furthermore, measurement settings are established such that the Bell inequality is violated. We consider, therefore, the following question: What is a general method for experimentalists to see the conflict between local realism and the quantum predictions only from actually measured data?

In many cases we build a local realistic theory for the observed data. However, many such theories are artificial and are disproved if some principles of physics are taken into account. An example of such a principle is rotational invariance of the value of a correlation function — the fact that the value of a correlation function does not depend on the orientation of reference frames. Taking this additional requirement into account rules out local realistic theories even in situations in which standard Bell inequalities allow an explicit construction of such theories.\textsuperscript{21}

In this paper, we study the physical phenomenon presented in Ref. 21. Here, we present a method by using two Bell operators.\textsuperscript{22} To this end, only a two-setting and two-particle Bell experiment reproducible by local realistic theories is needed. Such a Bell experiment also reveals the conflict between Bell–Zukowski inequality and Bell–Mermin inequality in the sense that Bell–Zukowski inequality\textsuperscript{23} is violated.

Our thesis is as follows. Consider two-qubit states that, under specific settings, give correlation functions reproducible by specific local realistic theory. Imagine that \( N \) copies of the states can be distributed among \( 2N \) parties, in such a way that each pair of parties shares one copy of the state. The parties perform a Bell–Greenberger–Horne–Zeilinger (GHZ) \( 2N \)-particle experiment\textsuperscript{15-20} on their qubits. Each of the pairs of parties uses the measurement settings noted above. Bell–Mermin operator, \( B \), for their experiment does not show a violation of local realism. Nevertheless, we find another Bell operator, which differs from \( B \) by a numerical factor, that does show such a violation.

This phenomenon occurs when the system is in a mixed two-qubit state. We analyze the threshold visibility for two-particle interference to reveal the conflict mentioned above. The threshold visibility agrees with the value to obtain a violation of Bell–Zukowski inequality.

2. Experimental Situation

Consider two-qubit states:

\[
\rho_{a,b} = V |\psi\rangle \langle \psi| + (1 - V) \rho_{\text{noise}}, \quad 0 \leq V \leq 1 ,
\]  

(1)
where $|\psi\rangle$ is Bell state as $|\psi\rangle = \frac{1}{\sqrt{2}}(|+a;+b| - |a;-b|)$. $\rho_{\text{noise}} = \frac{1}{4}I$ is the random noise admixture. The value of $V$ is interpreted as the reduction factor of the interferometric contrast observed in the two-particle correlation experiment. The states $|\pm b\rangle$ are eigenstates of $z$-component Pauli observable $\sigma^z_k$ for the $k$th observer. Here $a$ and $b$ are the label of parties (say Alice and Bob). Then we have $\text{tr}[\rho_{a,b}\sigma^a_x\sigma^b_x] = 0$, $\text{tr}[\rho_{a,b}\sigma^a_y\sigma^b_y] = V$ and $\text{tr}[\rho_{a,b}\sigma^a_z\sigma^b_z] = V$. Here $\sigma^k_x$ and $\sigma^k_y$ are Pauli-spin operators for $x$-component and for $y$-component, respectively. This set of experimental correlation functions is described with the property that they are reproducible by two-setting local realistic theories. See the following relations along with the arguments in Refs. 13 and 14

\begin{align}
\text{tr}[\rho_{a,b}\sigma^a_x\sigma^b_x] - \text{tr}[\rho_{a,b}\sigma^a_y\sigma^b_y] + \text{tr}[\rho_{a,b}\sigma^a_z\sigma^b_z] &= 2V \leq 2, \\
\text{tr}[\rho_{a,b}\sigma^a_x\sigma^b_x] + \text{tr}[\rho_{a,b}\sigma^a_y\sigma^b_y] - \text{tr}[\rho_{a,b}\sigma^a_z\sigma^b_z] &= 0 \leq 2, \\
\text{tr}[\rho_{a,b}\sigma^a_x\sigma^b_x] + \text{tr}[\rho_{a,b}\sigma^a_y\sigma^b_y] + \text{tr}[\rho_{a,b}\sigma^a_z\sigma^b_z] &= 0 \leq 2, \\
\text{tr}[\rho_{a,b}\sigma^a_x\sigma^b_x] - \text{tr}[\rho_{a,b}\sigma^a_y\sigma^b_y] - \text{tr}[\rho_{a,b}\sigma^a_z\sigma^b_z] &= 2V \leq 2.
\end{align}

In the following section, we use this kind of experimental situations. Interestingly, these experimental correlation functions compute a violation of Bell–Zukowski inequality. In order to do so, we present Bell operator method for such experimental data to reveal that constructed local realistic theories are not rotationally invariant if $V > 2(2/\pi)^2 \approx 0.81$. Of course, such a conflict between Bell–Zukowski inequality and Bell–Mermin inequality is derived only from actually measured data which is modeled by two-setting local realistic theories.

3. Conflict Between Bell–Zukowski Inequality and Bell–Mermin Inequality

Let $N_{2N}$ be $\{1, 2, \ldots, 2N\}$. We assume that $N$ copies of the states introduced in the preceding section are distributed among $2N$ parties, in such a way that each pair of parties shares one copy of the state

$$\rho^\otimes N = \rho_{1,2} \otimes \rho_{3,4} \otimes \cdots \otimes \rho_{N-1,N}.$$  

We assume that spatially separated $2N$ observers perform measurements on each of the $2N$ particles. The decision processes for choosing measurement observables are spacelike separated.

We assume that a two-orthogonal-setting Bell–GHZ $2N$-particle correlation experiment is performed. We assume measurement observables such that

$$A_k = \sigma^k_x, \quad A_k = \sigma^k_y.$$  

Each of the pairs of parties uses measurement settings such that they check the condition (2). The given $2^{2N}$ correlation functions are described with the property that they are reproducible by two-setting local realistic theories.
Bell–Mermin operators $B_{N_{2N}}$ and $B'_{N_{2N}}$ (defined as follows) do not show any violation of local realism as shown below.

Let $f(x, y)$ denote the function $\frac{1}{\sqrt{2}} e^{-i\pi/4} (x + iy)$, $x, y \in \mathbb{R}$. $f(x, y)$ is invertible as $x = \Re f - \Im f$, $y = \Re f + \Im f$. Bell–Mermin operators $B_{N_{2N}}$ and $B'_{N_{2N}}$ are defined by $18-20, 24 \quad f(B_{N_{2N}}, B'_{N_{2N}}) = \otimes_{k=1}^{2N} f(A_k, A'_k)$. Bell–Mermin inequality is expressed as

$$|\langle B_{N_{2N}} \rangle| \leq 1, \quad |\langle B'_{N_{2N}} \rangle| \leq 1,$$

where $B_{N_{2N}}$ and $B'_{N_{2N}}$ are Bell–Mermin operators defined by

$$f(B_{N_{2N}}, B'_{N_{2N}}) = \otimes_{k=1}^{2N} f(A_k, A'_k).$$

We also define $B_\alpha$ for any subset $\alpha \subset N_{2N}$ by

$$f(B_\alpha, B'_\alpha) = \otimes_{k \in \alpha} f(A_k, A'_k).$$

It is easy to see that, when $\alpha, \beta(\subset N_{2N})$ are disjoint,

$$f(B_{\alpha \cup \beta}, B'_{\alpha \cup \beta}) = f(B_\alpha, B'_\alpha) \otimes f(B_\beta, B'_\beta),$$

which leads to the following equations,

$$B_{\alpha \cup \beta} = (1/2)B_\alpha \otimes (B_\beta + B'_\beta) + (1/2)B'_\alpha \otimes (B_\beta - B'_\beta),$$

$$B'_{\alpha \cup \beta} = (1/2)B'_\alpha \otimes (B_\beta + B'_\beta) + (1/2)B_\alpha \otimes (B_\beta - B'_\beta).$$

In specific operators $A_k, A'_k$ given in Eq. (4), where $\sigma^k_x = |+k\rangle\langle-k| + |k\rangle\langle-k|$ and $\sigma^k_y = -i|+k\rangle\langle-k| + i|k\rangle\langle-k|$, we have (cf. Ref. 25)

$$f(A_k, A'_k) = (e^{-i\frac{\pi}{4}}/\sqrt{2}) (\sigma^k_x + i\sigma^k_y)$$

$$= e^{-i\frac{\pi}{4}} \sqrt{2} |+k\rangle\langle-k|$$

and

$$f(B_{N_{2N}}, B'_{N_{2N}}) = \otimes_{k=1}^{2N} f(A_k, A'_k)$$

$$= e^{-i\frac{2\pi}{4}} 2^N \otimes_{k=1}^{2N} |+k\rangle\langle-k|$$

$$= e^{-i\frac{2\pi}{4}} 2^N |1^{\otimes 2N}\rangle\langle 1^{\otimes 2N}|.$$  \hspace{0.5cm} (11)

Hence we obtain

$$B_{N_{2N}} = 2^{(2N-1)/2} (|\Psi^+_0\rangle\langle \Psi^+_0| - |\Psi^+_0\rangle\langle \Psi^-_0|),$$

where $e^{-i\frac{2N-1}{4}} 1^{\otimes 2N} = |1^{\otimes 2N}\rangle$. Here the states $|\Psi^\pm_0\rangle$ are Greenberger–Horne–Zeilinger (GHZ) states,\hspace{0.5cm} i.e.

$$|\Psi^\pm_0\rangle = \frac{1}{\sqrt{2}} (|0^{\otimes 2N}\rangle \pm |1^{\otimes 2N}\rangle).$$  \hspace{0.5cm} (13)
Measurements on each of $2N$ particles enable them to obtain $2^{2N}$ correlation functions. They get an average value of specific Bell–Mermin operator given in Eq. (12). According to Eq. (9), we obtain

$$\langle B_{N_{2N}} \rangle = \langle B'_{N_{2N}} \rangle = \prod_{i=2}^{N} \langle B_{i-1,i} \rangle = V^N (\leq 1) .$$

(14)

Bell–Mermin operators, $B_{N_{2N}}$ and $B'_{N_{2N}}$, for their experiment do not show any violation of local realism as we have mentioned above.

Nevertheless, we find a $2N$-partite Bell operator, which we call Bell–Żukowski operator $Z_{2N}$, which differs from $B_{N_{2N}}$ only by a numerical factor, that does show such a violation. Bell–Żukowski operator $Z_{2N}$ is as (cf. Appendix A, Eq. (A.22))

$$Z_{2N} = \frac{1}{2} \left( \frac{\pi}{2} \right)^{2N} \left( |\Psi_0^+ \rangle \langle \Psi_0^+ | - |\Psi_0^- \rangle \langle \Psi_0^- | \right) .$$

(15)

We see that Bell–Mermin operator given in Eq. (12) is connected to Bell–Żukowski operator $Z_{2N}$ in the following relation:

$$Z_{2N} = \frac{1}{2} \left( \frac{\pi}{2} \right)^{2N} \frac{1}{2^{(2N-1)/2}} B_{N_{2N}} .$$

(16)

We see that the specific two setting Bell $2N$-particle experiment in question computes an average value of Bell–Żukowski operator $\langle Z_{2N} \rangle$ via an average value of $\langle B_{N_{2N}} \rangle$.

From Bell–Żukowski inequality $|\langle Z_{2N} \rangle| \leq 1$, we have a condition on the average value of Bell–Mermin operator $\langle B_{N_{2N}} \rangle$ which is written by

$$|\langle B_{N_{2N}} \rangle| \leq 2 \left( \frac{2}{\pi} \right)^{2N} 2^{(2N-1)/2} .$$

(17)

Bell–Żukowski inequality $|\langle Z_{2N} \rangle| \leq 1$ is derived under the assumption that there are predetermined “hidden” results of measurements for all directions in a rotation plane for the system in a state. On the other hand, Bell–Mermin inequality is derived under the assumption that there are predetermined “hidden” results of measurements for two directions for the system in a state. Bell–Żukowski inequality governs rotationally invariant descriptions while Bell–Mermin inequality does not.

When $N \geq 2$ and $V$ is given by

$$2 \left( \frac{2}{\pi} \right)^{2N} 2^{(2N-1)/2} < V \leq 1 ,$$

(18)

we compute a violation of Bell–Żukowski inequality $|\langle Z_{2N} \rangle| \leq 1$, i.e. the explicit local realistic theories are not rotationally invariant. The condition (18) says that the threshold visibility decreases when the number of copies $N$ increases. In an extreme situation, when $N \to \infty$, we have the desired condition $V > 2(2/\pi)^2$
to show the conflict in question. It agrees with the value to get a violation of Bell–Zukowski inequality. (It also agrees with the value to get a violation of the generalized Bell inequality presented in Ref. 21.)

Thus given example by using two-qubit states reveals a violation of Bell–Zukowski inequality. The interesting point is that all the information to get a violation of Bell–Zukowski inequality is obtained only by a two-setting and two-particle Bell experiment reproducible by two-setting local realistic theories.

It presents a quantum-state measurement situation that admits local realistic theories for the given apparatus settings, but no local realistic theories which are rotationally invariant, even though the experiment is ruled by rotationally invariant laws. There is no local realistic theory for the experiment as a whole and so such a theory is only possible for certain settings.

What the thesis illustrates is that there is a further division among the measurement settings, those that admit rotational invariant local realistic theories and those that do not. This is another manifestation of the underlying contextual nature of realistic theories of quantum experiments.

4. Summary

In summary, we have presented a Bell operator method. This approach provides a means to check if an explicit theory is rotationally invariant, i.e. if a conflict between Bell–Zukowski inequality and Bell–Mermin inequality occurs. Our argument relies only on a two-setting and two-particle Bell experiment reproducible by a local realistic theory which is not rotationally invariant. Given a two-setting and two-particle Bell experiment reproducible by a specific local realistic theory, we compute a violation of Bell–Zukowski inequality. Measured data indicates that the explicit local realistic theories are not rotationally invariant. The conflict between Bell–Zukowski inequality and Bell–Mermin inequality is revealed.

This phenomenon occurs when the system is in a mixed state. We also analyze the threshold visibility for two-particle interference in order to bring about the phenomenon. The threshold visibility agrees well with the value to obtain a violation of Bell–Zukowski inequality.

Appendix A. Bell–Zukowski Inequality

Let us review Bell–Zukowski inequality proposed in Ref. 23. Let \( L(H) \) be the space of Hermitian operators acting on a finite-dimensional Hilbert space \( H \), and \( T(H) \) be the space of density operators acting on the Hilbert space \( H \). Namely, \( T(H) = \{ \rho | \rho \in L(H) \land \rho \geq 0 \land \text{tr}(\rho) = 1 \} \). We also consider a classical probability space \( (\Omega, \Sigma, M_\rho) \), where \( \Omega \) is a nonempty space, \( \Sigma \) is a \( \sigma \)-algebra of subsets of \( \Omega \), and \( M_\rho \) is a \( \sigma \)-additive normalized measure on \( \Sigma \) such that \( M_\rho(\Omega) = 1 \). The subscript \( \rho \) expresses the following meaning. The probability measure \( M_\rho \) is determined uniquely when a state \( \rho \) is specified.
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Consider a quantum state $\rho$ in $T(\bigotimes_{k=1}^{n} H_k)$, where $H_k$ represents a Hilbert space with respect to a party $k \in \mathbb{N}_n(=\{1, 2, \ldots, n\})$. We define measurable functions $f_k : o_k, \omega \mapsto f_k(o_k, \omega) \in [I(o_k), S(o_k)], o_k \in L(H_k), \omega \in \Omega$. Here $S(o_k)$ and $I(o_k)$ are the supremum and the infimum of the spectrum of $o_k \in L(H_k)$, respectively. These functions $f_k(o_k, \omega)$ do not depend on the choices of $v$'s on the other sites in $\mathbb{N}_n \setminus \{k\}$. By using the functions $f_k$, we define a quantum correlation function which admits a local realistic theory.

**Definition A.1.** A quantum correlation function $\text{tr}[\rho \otimes_{k=1}^{n} o_k]$ is said to admit a local realistic theory if and only if there exist a classical probability space $(\Omega, \Sigma, M_{\rho})$ and a set of functions $f_1, f_2, \ldots, f_n$, such that

$$\int_{\Omega} M_{\rho}(d\omega) \prod_{k=1}^{n} f_k(o_k, \omega) = \text{tr}[\rho \otimes_{k=1}^{n} o_k]$$

for a Hermitian operator $\otimes_{k=1}^{n} o_k$, where $o_k \in L(H_k)$. Note that there are several (noncommuting) observables per site, however the above definition is available for just one $o_k$ per site.

We consider a situation where each of the $n$ spatially separated observers has an infinite number of settings of measurements (in the $xy$ plane) to choose from. The operation of each of the measuring apparatuses is controlled by a knob. The knob sets a parameter $\phi$. An apparatus performs measurement of a Hermitian operator $\sigma_{\phi}$ on two-dimensional space with two eigenvalues $\pm 1$. The corresponding eigenstates are defined as $|\pm, \phi\rangle = (1/\sqrt{2})(|1\rangle \pm e^{i\phi}|0\rangle)$. The local phases that they are allowed to set are chosen as $0 \leq \phi^k < \pi$ for the $k$th observer. Bell–Zukowski inequality is written as

$$|\langle Z_n \rangle| \leq 1,$$

where the corresponding Bell operator $Z_n$ is

$$Z_n = \left( \frac{1}{2^n} \right) \int_{0}^{\pi} d\phi^1 \cdots \int_{0}^{\pi} d\phi^n \cos \left( \sum_{k=1}^{n} \phi^k \right) \otimes_{k=1}^{n} \sigma_{\phi^k},$$

where

$$\sigma_{\phi^k} = e^{-i\phi^k}|1^k\rangle\langle 0^k| + e^{i\phi^k}|0^k\rangle\langle 1^k|, \quad k \in \mathbb{N}_n.$$  

Bell–Zukowski operator $Z_n$ is a sum of an infinite number of Hermitian operators, except for the fixed number $1/(2^n)$. We mention why $Z_n$ given in Eq. (A.3) is a Bell operator when Eq. (A.2) is a Bell inequality as follows.

We assume that all of quantum correlation functions (every setting lies in $xy$ plane) admit a local realistic theory. Here each party $k$ performs locally measurements on an arbitrary single state $\rho$. 


Then, according to Definition A.1 (Eq. (A.1)), there exists a classical probability space \((\Omega, \Sigma, M_\rho)\) related to the state in question \(\rho\). And there exists a set of functions \(f_1, f_2, \ldots, f_n(\in [-1, 1])\) such that

\[
\int_\Omega M_\rho(d\omega) \prod_{k=1}^n f_k(\sigma_{\phi^k}, \omega) = \text{tr}[\rho \otimes_{k=1}^n \sigma_{\phi^k}]
\]

for every \(0 \leq \phi^k < \pi, \ k \in \mathbb{N}_n\). Hence an expectation of a sum of an infinite number of Hermitian operators (i.e. \(2^n Z_n\)) is bounded by the possible values of

\[
S_{\omega}^{(\infty, n)} = \int_0^\pi d\phi^1 \cdots \int_0^\pi d\phi^n \left[ \cos \left( \sum_{k=1}^n \phi^k \right) \prod_{k=1}^n f_k(\sigma_{\phi^k}, \omega) \right]
\]

\[
= \Re \left( \prod_{k=1}^n z^*_k \right),
\]

where \(z^*_k = \int_0^\pi d\phi^k f_k(\sigma_{\phi^k}, \omega) \exp(i\phi^k).

Let us derive an upper bound of \(S_{\omega}^{(\infty, n)}\). We may assume \(f_k = \pm 1\). Let us analyze the structure of the following integral

\[
z'_k = \int_0^\pi d\phi^k f_k(\sigma_{\phi^k}, \omega) \exp(i\phi^k)
\]

\[
= \int_0^\pi d\phi^k f_k(\sigma_{\phi^k}, \omega)(\cos \phi^k + i \sin \phi^k).
\]

Note that Eq. (A.7) is a sum of the following integrals:

\[
\int_0^\pi d\phi^k f_k(\phi^k, \omega) \cos \phi^k
\]

and

\[
\int_0^\pi d\phi^k f_k(\phi^k, \omega) \sin \phi^k.
\]

We deal here with integrals, or rather scalar products of \(f_k(\phi^k, \omega)\) with two orthogonal functions. We have

\[
\int_0^\pi d\phi^k \cos \phi^k \sin \phi^k = 0.
\]

The normalized functions \(\frac{1}{\sqrt{\pi/2}} \cos \phi^k\) and \(\frac{1}{\sqrt{\pi/2}} \sin \phi^k\) form a basis of a real two-dimensional functional space, which we call \(S^{(2)}\). Any function in \(S^{(2)}\) is of the form

\[
A \frac{1}{\sqrt{\pi/2}} \cos \phi^k + B \frac{1}{\sqrt{\pi/2}} \sin \phi^k,
\]

where \(A\) and \(B\) are constants, and that any normalized function in \(S^{(2)}\) is given by

\[
\cos \psi \frac{1}{\sqrt{\pi/2}} \cos \phi^k + \sin \psi \frac{1}{\sqrt{\pi/2}} \sin \phi^k = \frac{1}{\sqrt{\pi/2}} \cos(\phi^k - \psi).
\]
The norm \( \| f_k \| \) of the projection of \( f_k \) into the space \( S^{(2)} \) is given by the maximal possible value of the scalar product \( f_k \) with any normalized function belonging to \( S^{(2)} \), i.e.

\[
\| f_k \| = \max_{\psi} \int_0^\pi d\phi^k f_k(\phi^k, \omega) \frac{1}{\sqrt{\pi/2}} \cos(\phi^k - \psi).
\] (A.13)

As \( |f_k(\phi^k, \omega)| = 1 \), we have \( \| f_k \| \leq 2/\sqrt{\pi/2} \). Since \( \frac{1}{\sqrt{\pi/2}} \cos \phi^k \) and \( \frac{1}{\sqrt{\pi/2}} \sin \phi^k \) are two orthogonal basis functions in \( S^{(2)} \), we have

\[
\int_0^\pi d\phi^k f_k(\phi^k, \omega) \frac{1}{\sqrt{\pi/2}} \cos \phi^k = \cos \beta_k \| f_k \|
\] (A.14)

and

\[
\int_0^\pi d\phi^k f_k(\phi^k, \omega) \frac{1}{\sqrt{\pi/2}} \sin \phi^k = \sin \beta_k \| f_k \|,
\] (A.15)

where \( \beta_k \) is some angle. By using this fact, we put the value of (A.7) into the following form:

\[
z_k = \sqrt{\pi/2} \| f_k \| (\cos \beta_k + i \sin \beta_k)
= \sqrt{\pi/2} \| f_k \| \exp(i \beta_k).
\] (A.16)

Therefore, since \( \| f_k \| \leq 2/\sqrt{\pi/2} \), the maximal value of \( |z_k| \) is 2. Hence, we have \( |\prod_{k=1}^n z_k| \leq 2^n \). Then we get

\[
|S^{(\infty,n)}_\omega| \leq 2^n.
\] (A.17)

Let \( E(\cdot) \) represent an expectation on the classical probability space. If we integrate this relation (A.17) under normalized measure \( M_\omega(d\omega) \) over a space \( \Omega \), we obtain the relation (A.2). Here we use the relation that \( E(S^{(\infty,n)} \omega) = 2^n \text{tr} [\rho Z_n] \) (see Eq. (A.5)). Therefore, we prove Bell– Żukowski inequality (A.2) from an assumption. The assumption is that all of infinite number of quantum correlation functions (every setting lies in \( xy \) plane) admit a local realistic theory.

Let us consider matrix elements of Bell– Żukowski operator \( Z_n \) as given in Eq. (A.3) by using GHZ basis

\[
|\Psi^{\pm}_j \rangle = \frac{1}{\sqrt{2}} (|j\rangle|0\rangle \pm |2^{n-1} - j - 1\rangle|1\rangle),
\] (A.18)

where \( j = j_1 j_2 \cdots j_{n-1} \) is understood in binary notation. It is clear that no off-diagonal element appears, because of the form of the operator \( \sigma^k \) as given in Eq. (A.4).

Let \( \beta \) be a subset \( \beta \subset N_n \) and \( l(\beta) \) be an integer \( l_1 \cdots l_n \) in the binary notation with \( l_m = 1 \) for \( m \in \beta \) and \( l_m = 0 \) otherwise. And let \( j(\beta) \) be an integer binary-represented by \( l_1 \cdots l_{n-1} \). Then we define a two-to-one function \( g : \beta \mapsto g(\beta) \in \{0\} \cup N_{2^{(n-1)-1}} \) where \( g(\beta) \) takes the values \( j(\beta) \) and \( 2^{n-1} - j(\beta) - 1 \), respectively, for even and odd values of \( l(\beta) \).
In what follows, we show that \( h \equiv 0 \) for any subset \( N \subset N_n \). We also show that \( h \equiv 1 \) when \( N = \emptyset \) or \( \alpha = N_n \). When \( N \neq \emptyset \) or \( \alpha = N_n \), we have

\[
2^n \langle \Psi_0^+ | Z_n | \Psi_0^+ \rangle = \pm \frac{1}{2} \int_0^n \! d\phi_1 \cdots \int_0^n \! d\phi_n \cos^2 \left( \sum_{k=1}^n \phi_k \right) \\
= \pm \frac{1}{2} \int_0^n \! d\phi_1 \cdots \int_0^n \! d\phi_n \left[ 1 + \cos \left( 2 \sum_{k=1}^n \phi_k \right) \right] \\
= \pm \frac{1}{2} \Re \left\{ \int_0^n \! d\phi_1 \cdots \int_0^n \! d\phi_n \left[ 1 + \exp \left( 2i \sum_{k=1}^n \phi_k \right) \right] \right\} \\
= \pm \frac{\pi^n}{2} \pm \frac{1}{2} \Re \left( \prod_{k=1}^n \int_0^n \! d\phi_k \exp(2i\phi_k) \right). \tag{A.19}
\]

Since \( \int_0^n \! d\phi_k \exp(2i\phi_k) = 0 \), \( k \in N_n \), the last term vanishes. Hence we have

\[
\langle \Psi_0^+ | Z_n | \Psi_0^+ \rangle = \mp \frac{1}{2} \left( \frac{\pi}{2} \right)^n. \tag{A.20}
\]

On the other hand, when \( \alpha \neq \emptyset \), \( N_n \), we have

\[
2^n |\langle \Phi_{g(\alpha)}^\pm | Z_n | \Phi_{g(\alpha)}^\pm \rangle| = \int_0^n \! d\phi_1 \cdots \int_0^n \! d\phi_n \cos \left( \sum_{k \in \alpha} \phi_k - \sum_{k \in N_n \setminus \alpha} \phi_k \right) \\
\times \cos \left( \sum_{k \in \alpha} \phi_k - \sum_{k \in N_n \setminus \alpha} \phi_k \right) \\
= \frac{1}{2} \int_0^n \! d\phi_1 \cdots \int_0^n \! d\phi_n \left[ \cos \left( 2 \sum_{k \in \alpha} \phi_k \right) + \cos \left( 2 \sum_{k \in N_n \setminus \alpha} \phi_k \right) \right] \\
= \frac{\pi^{|N_n \setminus \alpha|}}{2} \Re \left( \prod_{k \in \alpha} \int_0^n \! d\phi_k \exp(2i\phi_k) \right) \\
+ \frac{\pi^{|\alpha|}}{2} \Re \left( \prod_{k \in N_n \setminus \alpha} \int_0^n \! d\phi_k \exp(2i\phi_k) \right). \tag{A.21}
\]

Since \( \int_0^n \! d\phi_k \exp(2i\phi_k) = 0 \), \( k \in N_n \), the last two terms vanish.

Hence, Bell operator \( Z_n \) as given in Eq. (A.3) is rewritten as

\[
Z_n = \frac{1}{2} \left( \frac{\pi}{2} \right)^n (|\Psi_0^+ \rangle \langle \Psi_0^+ | - |\Psi_0^- \rangle \langle \Psi_0^- |). \tag{A.22}
\]
Acknowledgments

This work has been supported by Frontier Basic Research Programs at KAIST and K.N. is supported by a BK21 research grant.

References

1. J. S. Bell, Physics 1, 195 (1964).